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The role of coherence entropy of physical twin observables in entanglement

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Abstract
The concept of physical twin observables (PTO) for bipartite quantum states, introduced and proved relevant for quantum information theory in recent work, is substantially simplified. The relation of observable and state is studied in detail from the point of view of coherence entropy. Properties of this quantity are further explored. It is shown that, besides for pure states, also for a class of mixed states, quantum discord (measure of entanglement) can be expressed through the coherence entropy of a PTO complete in relation to the state.

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1. Introduction

This study hinges on two concepts, that of coherence entropy and that of twin observables. To understand what *coherence* is, one starts with the lack of it. One considers a quantum state (density operator) ρ and a discrete observable (Hermitian operator) in spectral form with distinct eigenvalues $A = \sum_l a_l P_l$ (possible $a_l = 0$ included) because one deals with a relative concept: observable in relation to state. Let B be another observable. Its average in ρ is

$$\langle B\rangle = \mathrm{Tr}(\rho B) = \sum_l \mathrm{Tr}(P_l \rho B). \tag{1}$$

The question is when is this a mixture of separate contributions from the eigenvalues a_l of A, i.e.

$$\sum_l \mathrm{Tr}[(P_l \rho P_l)B] = \sum_l w_l \langle B\rangle_l \tag{2a}$$

where

$$\forall l: \quad w_l \equiv \mathrm{Tr}(P_l \rho P_l) \tag{2b}$$

are the statistical weights, and, for $w_l > 0$,

$$\langle B\rangle_l \equiv \mathrm{Tr}[(P_l \rho P_l / w_l)B]. \tag{2c}$$

It is easy to convince oneself that putting the analogous question in terms of the classical counterparts, the answer is 'always'. One says that in classical physics there is no coherence, i.e. coherence is an unknown concept there.

Further, since $[A, B] = 0$ implies $\forall l: \; [P_l, B] = 0$, it is easy to see that (2*a*) is valid in this case. One says that coherence never shows up with respect to an observable B compatible (commuting) with the observable A at issue.

To put the above question in a more specific form, we ask when (1) is equal to (2*a*) for *all* observables B that are *incompatible* with A. One can prove that this is the case if and only if $[A, \rho] = 0$, i.e. in the case of *compatibility of observable and state* [1].

Thus, if A and ρ are *incompatible*, and only then, (1) is not equal to (2*a*) for all B, and this is called *coherence* of A in relation to ρ.

Experimentally coherence is usually observed as *interference*, i.e. a cooperative contribution of two or more eigenevents (eigenprojectors) P_l of A in the average of some observable B incompatible with A. (Note that in the case of events, average and probability are the same thing.)

The best-known example of interference is that on two slits. In a previous paper [2], I have given a detailed description of it along the lines of this introduction (with the additional intricacy of an evolution between passage of the two-slit screen and arrival at the detection screen).

The next question is how to find a quantity that would be *the amount of coherence* of A in ρ. On intuitive grounds one can say that it must satisfy three requirements.

(i) It must be a function of A and ρ.
(ii) In the case of compatibility $[A, \rho] = 0$ it must be zero, otherwise it must be positive. Some of the eigenevents P_l of A may be compatible with ρ, hence part of the average of B may be expressible as an average of separate contributions. This part is irrelevant for coherence, because the latter is negation of such a mixture. Hence, the third requirement is
(iii) The desired quantity should depend only on those eigenevents P_l of A that are not compatible with ρ, and not at all on those that are.

In a previous paper [3], the *amount of coherence* of A in ρ was denoted by $E_C(A, \rho)$ and defined as

$$E_C(A, \rho) \equiv S\left(\sum_l P_l \rho P_l\right) - S(\rho) \tag{3}$$

where $S(\rho)$ is the von Neumann entropy of the quantum state ρ. The quantity $E_C(A, \rho)$ is the (nonnegative) entropy increase in ideal measurement of A in ρ. It is called *the coherence entropy*. It satisfies the first two intuitive requirements. It also satisfies the third one as proved below (theorem 2).

Physical twin observables (PTO) were shown to be relevant [3] for important questions in quantum information theory [4, 5]. In particular, PTO can be made use of both for defining the quasi-classical or subsystem-measurement-accessible part and the purely quantum part, i.e. the amount of entanglement or the quantum discord [6] in a general bipartite pure state.

The definition of PTO applies to two opposite-subsystem observables (Hermitian operators) A_1 and A_2 that have a special relation to a given composite (1 + 2)-system state (density operator) ρ_{12}. The definition in [3] begins with the (very strong) requirement of *compatibility* (commutation) of the observables (operators) with the corresponding subsystem states (reduced density operators):

$$[A_s, \rho_s] = 0 \qquad s = 1, 2 \tag{4}$$

where, of course, the subsystem states are $\rho_s \equiv \mathrm{Tr}_{s'}\rho_{12}$, $s, s' = 1, 2$, $s \neq s'$, and '$\mathrm{Tr}_{s'}$' denotes the partial trace over subsystem s'. Further, a *bijection* between the *detectable eigenvalues* of A_1 and A_2 is required such that if P_s^i are the eigenprojectors corresponding to the *i*th detectable pair of eigenvalues, $s = 1, 2$, connected by the bijection, then the following so-called *algebraic condition*

$$\forall i: \quad P_1^i \rho_{12} = P_2^i \rho_{12} \tag{5a}$$

has to be satisfied. An equivalent condition is the measurement-theoretic one claiming that

$$\forall i: \quad P_1^i \rho_{12} P_1^i = P_2^i \rho_{12} P_2^i. \tag{5b}$$

(There are two more equivalent conditions [3] that will not be needed in this paper.)

In section 2 it is demonstrated that the expounded definition of PTO can be (substantially) simplified. In section 3 the concept of coherence entropy is studied by clarifying the basic necessary property: that the given observable should be 'discrete in relation to' the given state. Further, some well-known entropy inequalities are put in the form of an equality ((16) below) and displayed on a diagram (see figure 1). In section 4 the part of the spectrum of the observable that is actually responsible for determining the coherence entropy is singled out. Thus, the third intuitive requirement for coherence entropy is shown to be valid. In section 5 the partial order 'finer–coarser in relation to' among observables is studied and the concept of 'complete in relation to' a given state is investigated.

In section 6 incompatibility of observable and state that is exclusively due to quantum correlations is discussed. In section 7, a class of mixed states is identified in which the coherence entropy of physical twin observables can be viewed as constituting the quantum discord (the entire amount of entanglement). In section 8 the main results are summed up.

2. Redundancy of compatibility as a requirement

Let ρ_{12} be a bipartite state. We call the eigenvalues of observables that have positive probability in ρ_{12} detectable ones.

Theorem 1. *Let A_1 and A_2 be opposite-subsystem observables, and let there exist a bijection between all detectable eigenvalues of A_1 and all those of A_2 such that, upon using the common index i for the pair of corresponding eigenvalues, the algebraic condition (5a) is valid. Finally, let the total probability of detectable eigenvalues of A_1 and separately of A_2 be 1. Then the compatibility (4) is a consequence.*

In proving the theorem we use the known lemma stating the following equivalence between two expressions of one and the same elementary relation between an event (projector) P and a state ρ expressing certainty:

$$\mathrm{Tr}\,\rho P = 1 \quad \Leftrightarrow \quad P\rho = \rho \tag{6}$$

(cf [7] if proof is desired).

Proof. Let $\{a_i^s : \forall i\}$, $s = 1, 2$ denote the detectable eigenvalues of A_1 and of A_2, respectively. Let, further, $\{P_s^i : \forall i\}$ be the corresponding eigenprojectors. (We refer to them as detectable ones.) Finally, let $P_s \equiv \sum_i P_s^i$, $s = 1, 2$. We write the observables in the form

$$A_s = \sum_i a_i^s P_s^i + P_s^\perp A_s P_s^\perp \qquad s = 1, 2 \tag{7}$$

where $P_s^\perp$ denotes the orthocomplementary projector of P_s.

To prove commutation of the undetectable parts with ρ_s, we utilize the second relation in (6):

$$\big(P_s^{\perp} A_s P_s^{\perp}\big)\rho_s = \big(P_s^{\perp} A_s P_s^{\perp}\big)(P_s \rho_s)$$

hence

$$\big(P_s^{\perp} A_s P_s^{\perp}\big)\rho_s = 0 = \rho_s\big(P_s^{\perp} A_s P_s^{\perp}\big).$$

(The last equality is due to adjoining the preceding one.)

Commutation of the detectable parts can be proved as follows: making use of (5*a*) and of its adjoint, one has

$$P_s^i \rho_s = P_s^i \,\mathrm{Tr}_{s'} \rho_{12} = \mathrm{Tr}_{s'}\big[P_s^i \rho_{12}\big] = \mathrm{Tr}_{s'}\big[P_{s'}^i \rho_{12}\big] = \mathrm{Tr}_{s'}\big[\rho_{12} P_{s'}^i\big] = \mathrm{Tr}_{s'}\big[\rho_{12} P_s^i\big] = (\mathrm{Tr}_{s'} \rho_{12}) P_s^i$$
$$= \rho_s P_s^i \qquad s, s' = 1, 2 \qquad s' \neq s.$$

Hence, in view of (7), the validity of (4) is established. □

Thus, one can say that two opposite-subsystem observables A_1 and A_2 are *physical twin observables* with respect to a bipartite state ρ_{12} if the conditions of theorem 1 are valid.

It may be the case that one has physical twin observables such that the detectable eigenvalues that correspond to each other via the mentioned bijection are *equal* for all values of i, then (5*a*) is replaceable by the stronger (and more concise) algebraic relation

$$A_1 \rho_{12} = A_2 \rho_{12}. \tag{8}$$

This relation by itself implies all the rest of the properties of the observables in relation to ρ_{12} [8, 9]. (Cf [3] for the properties not mentioned in this paper.)

It is noteworthy that a comparison of (8) and (5*a*) reveals that each two corresponding detectable eigenprojectors P_1^i, P_2^i satisfy the stronger algebraic relation (8). Hence, as proved in [9], they are compatible with the corresponding subsystem states, i.e. (4) is valid, *mutatis mutandis*, for them. Then, a glance at (7) makes it clear that for the validity of (4) one actually needs only to prove compatibility of the undetectable parts $\big(P_1^{\perp} A_1 P_1^{\perp}\big)$, $\big(P_2^{\perp} A_2 P_2^{\perp}\big)$ with the corresponding subsystem states (e.g., as done in the proof of the theorem).

In the special case characterized by (8), A_1, A_2 should be called *algebraic twin observables*. They were studied in detail in previous work [8, 9]. There such observables were called simply 'twin observables'. In a more recent investigation [3], this practical terminology was utilized for physical twin observables as it should be in view of the fact that the latter, being more general, can be expected to have a wider scope of application.

As was mentioned in the introduction, in a recent study [3] the amount of purely quantum correlation or entanglement (or quantum discord) of an arbitrary given pure state $|\Phi\rangle_{12}$ was shown to be 'carried' by a specially constructed pair of twin observables. The way the quantum discord is 'carried' is expressed via the notion of coherence entropy. This result requires generalization.

In the next three sections we ignore bipartite states and twin observables for the time being, and make a precise analysis of the concept of coherence entropy (to be able to apply it to PTO).

3. Observables that are discrete in relation to a state

Let us rewrite the spectral form of a given *discrete observable* $A = \sum_l a_l P_l$ (distinct eigenvalues) indexing the eigenvalues and eigenevents that are detectable in a given state ρ by i, and those that are undetectable by m:

$$A = \sum_i a_i P_i + \sum_m a_m P_m. \tag{9}$$

Lemma 1. *Only the detectable eigenevents contribute to the coherence entropy*

$$E_C(A,\rho) = E_C\left(\left(\sum_i a_i P_i\right), \rho\right) \tag{10}$$

and the event P defined as $P \equiv \sum_i P_i$ *is certain in* ρ.

Proof. It is easy to see (by using the orthocomplement) that the following claim is an equivalent form of the relations in (6): an event P is undetectable in a state ρ if and only if $P\rho = 0$. Hence, substituting 9 in (3), (10) immediately ensues. Further, for the same reason,

$$1 = \mathrm{Tr}\left[\left(\sum_l P_l\right)\rho\right] = \mathrm{Tr}\left[\left(\sum_i P_i\right)\rho\right].$$

□

Lemma 1 enables one to see how widely one can extend the set of all discrete observables to obtain the widest set of elements for which the coherence entropy concept is applicable. (This widest set was introduced in an *ad hoc* manner [3].)

Every observable A can be written in the (partly) spectral form

$$A = \sum_i a_i P_i + P^{\perp} A P^{\perp} \tag{11}$$

(the a_i are distinct detectable eigenvalues, possible $a_i = 0$ included; i enumerates all of them). Naturally, the singled out detectable discrete part (the first term on the RHS) may be zero.

Definition 1. *Let A be an observable and* ρ *a state such that the total probability of detectable eigenvalues of A in* ρ *is one. We say that A is* discrete in relation to ρ. *Further, we define the* coherence entropy $E_C(A,\rho)$ *for such an observable as*

$$E_C(A,\rho) \equiv E_C\left(\left(\sum_i a_i P_i\right), \rho\right). \tag{12}$$

(Note that on the RHS we have the known coherence entropy of a discrete observable, whereas on the LHS we define this for A that need not be discrete in the absolute sense.)

To understand what class of observables we are dealing with, we make some elaboration.

Lemma 2. *If a general observable A and a state* ρ *are given, the total probability of the detectable eigenvalues of the former in the latter is* one if and only if *enumerating all detectable eigenvalues by i, the projector P* ($\equiv \sum_i P_i$) *projects onto a subspace that* contains *the range of* ρ.

If Q denotes the range projector of ρ, then the algebraic form of the (geometric) characteristic condition in the lemma is

$$PQ = Q. \tag{13}$$

This is a further known characteristic condition (cf (6)) for an event being certain in a state. (For the reader's convenience, it is proved in appendix A.)

Lemma 3. *If* $[A,\rho] = 0$, *then also* $[A,Q] = 0$, *and the reducee* A' *of A in the range* $\mathcal{R}(Q)$ *is* discrete *(in the absolute sense). Its eigenvalues are precisely the detectable eigenvalues* $\{a_i : \forall i\}$ *of A. The observable A is discrete in relation to* ρ.

Proof. Since A commutes with all eigenprojectors of ρ, and $Q = \sum_k Q_k$ (sum of all eigenprojectors of ρ corresponding to positive eigenvalues), also $[A,Q] = 0$ is valid. On account of $\forall k: \ [A, Q_k] = 0$, A reduces in each eigensubspace $\mathcal{R}(Q_k)$ of ρ. Since these are finite dimensional (the positive eigenvalues of ρ add up to 1), A' is discrete. Since by definition $\forall i: \ 0 < \mathrm{Tr}(P_i\rho) = \mathrm{Tr}[(P_iQ)\rho]$, one has $P_iQ \neq 0$. Hence, the reducee of P_iQ in $\mathcal{R}(Q)$ is a nonzero projector. It is the eigenprojector of A' corresponding to the eigenvalue a_i. Thus, each detectable eigenvalue of A is an eigenvalue of A'. On the other hand, each eigenvalue of A' (it is, of course, also an eigenvalue of A) is detectable because the reducee of ρ in $\mathcal{R}(Q)$ is nonsingular (see appendix B for this implication). Finally, the sum $\sum_i \mathrm{Tr}\, P_i\rho = \sum_i \mathrm{Tr}[(P_iQ)\rho]$ must be 1 because, designating by primes the reducees in $\mathcal{R}(Q)$, one has $\mathrm{Tr}[(P_iQ)\rho] = \mathrm{Tr}[(P_iQ)'\rho']$ and $\sum_i \mathrm{Tr}[(P_iQ)'\rho'] = \mathrm{Tr}\,\rho' = \mathrm{Tr}\,\rho = 1$. Hence, A is discrete in relation to ρ as claimed. □

Henceforth and throughout this paper when a state is given by an observable we mean one that is discrete in relation to the state (or otherwise we will say that we have a more general observable).

In general, the spectral part $P^{\perp}AP^{\perp}$ (see (11)) may contain a continuous spectrum if the null space of ρ is infinite dimensional. (If the null space is finite dimensional, then the observable must be discrete in the absolute sense). In the rest of this section we expound a basic relation of the coherence entropy $E_C(A, \rho)$ to its 'neighbouring' entropies. It is an immediate consequence of the following fundamental inequalities. (They have become of classical value in quantum entropy theory.) Let $\forall i: p_i \equiv \mathrm{Tr}(P_i\rho)$. Then [10]

$$\sum_i p_i S(P_i\rho P_i/p_i) \leqslant S(\rho) \tag{14a}$$

$$S(\rho) \leqslant S\left(\sum_i P_i\rho P_i\right). \tag{14b}$$

One has equality in the first inequality if and only if $\forall i: S(P_i\rho P_i/p_i) = S(\rho)$. The second inequality reduces to an equality if and only if one has compatibility: $\forall i: \ [P_i, \rho] = 0$.

Since $\sum_i P_i\rho P_i$ is an orthogonal mixture, the mixing property of entropy [11] applies

$$S\left(\sum_i P_i\rho P_i\right) = H(p_i) + \sum_i p_i S(P_i\rho P_i/p_i) \tag{15a}$$

where

$$H(p_i) \equiv -\sum_i p_i \ln p_i \tag{15b}$$

is the Shannon entropy of the probability distribution $\{p_i : \forall i\}$. It is often called the mixing entropy. But for our purposes we view it as the *entropy of the observable* A in ρ, and denote it by $S(A, \rho)$. This is a well-known concept (cf, e.g., [12]). It equals the amount of information that one can gain about A when measuring it in ρ.

We can thus rewrite (15*a*) as

$$S(A, \rho) = E_C(A, \rho) + \left(S(\rho) - \sum_i p_i S(P_i\rho P_i/p_i)\right). \tag{16}$$

It is noteworthy that $E_C(A, \rho)$ is nonnegative on account of (14*b*), and the second term on the RHS of (16) is also nonnegative due to (14*a*). (Relation (16) is the same as relation (20) in [3].)

It is obvious from (16) that

$$E_C(A, \rho) \leqslant S(A, \rho). \tag{17}$$

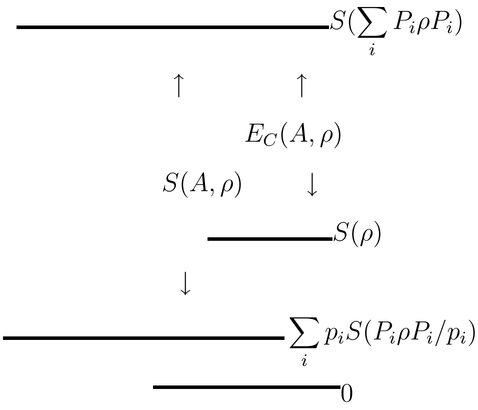

Figure 1. Entropy level diagram.

In other words, when measuring A in ρ, the amount of obtainable information on A is larger or equal to the entropic 'price' that one has to pay (for measuring an incompatible observable).

In case of compatibility $[A, \rho] = 0$, (16) reduces to

$$S(A, \rho) = S(\rho) - \sum_i p_i S(P_i \rho P_i / p_i).$$

Since this is the mixing property of entropy applied to the orthogonal decomposition $\rho = \sum_i P_i \rho P_i$, one has $S(A, \rho) = 0$ if and only if $\forall i : P_i \rho P_i / p_i = \rho$ (a strengthened from of the characteristic conditions for equality in (14*a*)).

Equality (16) is displayed in the self-explanatory *entropy level diagram* (see figure 1) with vertical distances representing the entropy quantities at issue.

4. The spectral part of the observable that determines the coherence entropy

Let a state ρ be given, and let us consider the detectable eigenprojectors P_i of a given observable A. Some of these may commute with ρ. Following the terminology used in [13], we call the eigenprojectors that do not commute with ρ *weak* ones; those that do commute, we call *strong* ones. For further analysis we enumerate the weak eigenprojectors by j, and the strong ones by k. Further, we write the spectral form of A as consisting, in general, of a *weak component observable* A_w, a *strong component observable* A_{st} and the irrelevant part with respect to ρ:

$$A = A_w + A_{st} + P^{\perp} A P^{\perp} \tag{18a}$$

where

$$A_w \equiv \sum_j a_j P_j \tag{18b}$$

and

$$A_{st} \equiv \sum_k a_k P_k. \tag{18c}$$

Let us further define the *weak probability* p_w, and, for the case when it is nonzero, the *weak component state* ρ_w

$$p_w \equiv \mathrm{Tr}\left[\left(\sum_j P_j\right)\rho\right] \tag{19a}$$

$$\rho_w \equiv \left(\sum_j P_j\right) \rho \left(\sum_j P_j\right) \Big/ p_w. \tag{19b}$$

Then we have the following *orthogonal decomposition* of the state:

$$\rho = p_w \rho_w + (1 - p_w)\rho_{st} \tag{20}$$

where the *strong component state* is given by the *orthogonal mixture of states*

$$\rho_{st} \equiv \sum_k \{[p_k/(1 - p_w)]\rho_k\} \tag{21a}$$

and $\forall k$:

$$p_k \equiv \mathrm{Tr}(P_k \rho) \tag{21b}$$

$$\rho_k \equiv P_k \rho P_k / p_k = P_k \rho / p_k. \tag{21c}$$

Definition 2. *In the three cases when* $p_w = 0$, $p_w = 1$ *and* $0 < p_w < 1$ *we say that the observable* A *is strong, weak and intermediary, respectively, regarding* ρ.

If ρ is a pure state, then every observable is weak with respect to it (with the exception of a constant multiplying a projector that does not change this state vector). If, on the other hand, one has $[A, \rho] = 0$, which is equivalent to the property that all detectable eigenprojectors of A are strong, then A is strong.

Remark 1. The eigenprojectors P_k are called strong because each of them 'cuts' a separate component state ρ_k 'out of' ρ (cf (21*c*)), whereas the weak eigenprojectors P_j 'cut out' a component state only in cooperation with all of them (cf (19*b*)).

Theorem 2. *The coherence entropy of* A *is equal to the coherence entropy of the weak component observable in the weak component state multiplied by the weak probability:*

$$E_C(A, \rho) = p_w E_C(A_w, \rho_w). \tag{22}$$

Proof. Definition (12) with (3) and (20) with definitions (19*a*), (19*b*) and (21*a*)–(21*c*) imply

$$\mathrm{LHS}(22) = S\left(p_w \sum_i (P_i \rho_w P_i) + (1 - p_w) \sum_i (P_i \rho_{st} P_i)\right) - S\,(p_w \rho_w + (1 - p_w)\rho_{st})$$
$$= S\left(p_w \sum_j (P_j \rho_w P_j) + (1 - p_w)\rho_{st}\right) - S(p_w \rho_w + (1 - p_w)\rho_{st}).$$

Since both entropies are taken of orthogonal state decompositions, one can apply the mixing property of entropy, and obtain

$$\mathrm{LHS}(22) = \left[H(p_w) + p_w S\left(\sum_j P_j \rho_w P_j\right) + (1 - p_w) S(\rho_{st})\right]$$
$$- [H(p_w) + p_w S(\rho_w) + (1 - p_w) S(\rho_{st})].$$

($H(p_w)$ being the Shannon entropy of the probability distribution $\{p_w, (1 - p_w)\}$). After cancellations the RHS(22) is obtained. □

5. Finer observables and complete ones in relation to a state

Now we turn to a special relation of two observables with respect to a given state.

Definition 3. *Let ρ be a given state, and let A and A' be two observables. Let further the detectable eigenprojectors of A' further decompose those of A:*

$$\forall i: \quad P_i = \sum_{i'} P_{i,i'} \tag{23}$$

where $\{P_{i,i'} : \forall i, i'\}$ are eigenprojectors of A' corresponding to its distinct eigenvalues $\{a'_{i,i'} : \forall i, i'\}$. Then A' is finer than or a refinement of A and the latter is coarser than or a coarsening of the former in relation to ρ and we write $A' \overset{\rho}{\geqslant} A$. If at least one of the sums in (23) is nontrivial in the sense that it has at least two detectable terms on the RHS, then A' is strictly finer than A etc in relation to ρ, and we write $A' \overset{\rho}{>} A$. Otherwise, A' and A are equal in relation to ρ, and we write $A' \overset{\rho}{=} A$.

Lemma 4. *If two observables A' and A are such that the former is a refinement of the latter in relation to the state ρ, then the entropy of the latter in ρ does not exceed that of the former*

$$A' \overset{\rho}{\geqslant} A \quad \Rightarrow \quad S(A', \rho) \geqslant S(A, \rho). \tag{24}$$

The entropies are equal if and only if $A' \overset{\rho}{=} A$.

Proof. Evidently, (23) implies $\forall i: \ p_i = \sum_{i'} p_{i,i'}$, where $p_i \equiv \mathrm{Tr}(P_i\rho)$ and $p_{i,i'} \equiv \mathrm{Tr}(P_{i,i'}\rho)$. One can write

$$p_{i,i'} = \sum_m p_m[\delta_{m,i}(p_{i,i'}/p_i)]$$

where m takes on the same values as i, and $p_m \equiv \mathrm{Tr}(P_m\rho)$. Denoting by $p^{(m)}_{i,i'}$ the probability distributions $[\delta_{m,i}(p_{i,i'}/p_i)]$ on the set of all pairs (i, i'), one has disjointness $p^{(m)}_{i,i'} p^{(m')}_{i,i'} = \delta_{m,m'}\big(p^{(m)}_{i,i'}\big)^2$. Hence, we are dealing with a disjoint decomposition of the probability distribution $p_{i,i'}$ (the classical discrete counterpart of an orthogonal decomposition of a quantum state), and we can apply the mixing property resulting in the following relation between the Shannon entropies (cf (15b)):

$$H(p_{i,i'}) = H(p_i) + \sum_m p_m H\big(p^{(m)}_{i,i'}\big).$$

On account of the definition of the entropy of an observable in a state (see the text beneath (15b)), the last relation can be written as

$$S(A', \rho) = S(A, \rho) + \sum_m p_m H\big(p^{(m)}_{i,i'}\big). \tag{25}$$

The nonnegativity of the last term bears out the first claim. If $A' \overset{\rho}{>} A$, at least one of the sums in (23) is nontrivial, e.g., for $i = m$. Then at least two pairs of indices (m, i'), (m, i'') $i' \neq i''$ enumerate detectable eigenvalues of A'; hence, also the decomposition of the corresponding probability is nontrivial, and $H\big(p^{(m)}_{i,i'}\big)$ is positive. This proves the second claim. □

Theorem 3. *If A' is a refinement of A in relation to a given state ρ, then the coherence entropy of the latter does not exceed that of the former in this state:*

$$E_C(A', \rho) \geqslant E_C(A, \rho). \tag{26}$$

The two entropies are equal if and only if the observable A' is compatible with the state $\sum_i P_i \rho P_i$, i.e.

$$\left[A', \sum_i P_i \rho P_i\right] = 0. \tag{27a}$$

The coherence entropy of A' is strictly larger than that of A (both in ρ) if and only if there exists a nontrivial sum in (23), and one has for the corresponding value of i:

$$\exists i' \neq i'' : \quad P_{i,i'} \rho P_{i,i''} \neq 0. \tag{27b}$$

Proof. Measurement of A' in an ideal way changes both ρ and the state $\sum_i P_i \rho P_i$ into one and the same state $\sum_i \sum_{i'} P_{i,i'} \rho P_{i,i'}$. Hence, $S\big(\sum_i \sum_{i'} P_{i,i'} \rho P_{i,i'}\big) \geqslant S\big(\sum_i P_i \rho P_i\big)$ (cf (14*b*)). Inequality (26) then follows (cf (12) and (3)). Criterion (27*a*) is a consequence of that for equality in (14*b*). The last claim follows from the facts that (27*a*) is equivalent to $\sum_i \sum_{i'} P_{i,i'} \rho P_{i,i'} = \sum_i P_i \rho P_i$, and that (27*b*) is the negation of this. □

Now we turn to exploring the last term in (16) with respect to comparison of A and a finer obseravable A' in relation to ρ. Though the probabilities p_i have to be positive due to the definition of the indices i, this is not the case with $p_{i,i'}$. If this probability is zero, the corresponding state and its entropy are not defined. But, for simplicity, we assume, as is usually done in such cases, that $p_{i,i'} S(P_{i,i'} \rho P_{i,i'}/p_{i,i'})$ is simply zero.

Lemma 5. *If A' is a refinement of A in ρ, then the entropy decrease, i.e. the second term on the RHS of (16), corresponding to A', is larger than or equal to that corresponding to A:*

$$\left\{S(\rho) - \sum_i \sum_{i'} [p_{i,i'} S(P_{i,i'} \rho P_{i,i'}/p_{i,i'})]\right\} \geqslant \left\{S(\rho) - \sum_i [p_i S(P_i \rho P_i/p_i)]\right\}. \tag{28}$$

One has equality if and only if

$$\forall i, i', \quad p_{i,i'} > 0 : \quad S(P_{i,i'} \rho P_{i,i'}/p_{i,i'}) = S(P_i \rho P_i/p_i).$$

This condition is satisfied when $A' \stackrel{\rho}{=} A$. But it may be valid also for $A' \stackrel{\rho}{>} A$.

Proof. One has

$$\mathrm{LHS}(28) - \mathrm{RHS}(28) = \sum_i \left[p_i \left[S(P_i \rho P_i/p_i) - \sum_{i'} ((p_{i,i'}/p_i) S[P_{i,i'}(P_i \rho P_i/p_i) P_{i,i'}/(p_{i,i'}/p_i)])\right]\right]$$

since $P_{i,i'} = P_{i,i'} P_i$ (cf (23)). Each term in the sum $\sum_i$ is nonnegative on account of (14*a*). The last claim follows from the equality conditions in (14*a*). □

Remark 2. If $A' \stackrel{\rho}{>} A$ but $[A', \sum_i P_i \rho P_i] = 0$, then $S(A', \rho) > S(A, \rho)$, but $E_C(A', \rho) = E_C(A, \rho)$. In this case the claimed criterion for equality in lemma 5, i.e. for lack of enlargement in the entropy decrease, is not valid. Namely, $\forall i : P_i \rho P_i/p_i = \sum_{i'} [(p_{i,i'}/p_i)(P_{i,i'} \rho P_{i,i'}/p_{i,i'})]$, and the average entropy in a mixture is always less than that of the mixture itself unless the mixture is trivial. And this is not the case for at least one value of i.

Definition 4. *If $A' \stackrel{\rho}{\geqslant} A$ implies $A' \stackrel{\rho}{=} A$, i.e. if A does not have a nontrivial refinement in relation to the given state ρ, then we say that A is* complete in relation to ρ.

Remark 3. Evidently, A is complete in relation to ρ if and only if further decomposition is not possible, i.e. for each detectable eigenprojector P_i of A there exists a state vector $|i\rangle$ such that

$$P_i \rho P_i / [\mathrm{Tr}(P_i \rho)] = |i\rangle\langle i|. \tag{29}$$

Lemma 6. *If an observable A and a state ρ in relation to which the former is discrete are* compatible, *i.e.* $[A, \rho] = 0$*, and if Q denotes the range projector of ρ, then the observable is* complete *in relation to the state* if and only if *for each value of i there exists a state $|i\rangle$ such that $P_i Q = |i\rangle\langle i|$.*

Proof. On account of the commutation, one can choose a common eigenbasis of A and ρ. Let us write its subbasis spanning $\mathcal{R}(Q)$ as $\{|i, k\rangle : \forall i, \forall k\}$, where $\forall i : \quad P_i|i, k\rangle = |i, k\rangle$. Denoting the corresponding eigenvalues of ρ by $\{r_{i,k} : \forall i, \forall k\}$, we can write the spectral form $P_i \rho P_i = \rho(P_i Q) = \sum_k r_{i,k}|i, k\rangle\langle i, k|$. It is now obvious that one has completeness (cf (29)) if and only if for each value of i there is only one value of k. Then $|i\rangle \equiv |i, k\rangle$. □

Remark 4. Since in the case of compatibility of A and ρ one has $\mathcal{R}(P_i Q) = \mathcal{R}(P_i) \cap \mathcal{R}(Q)$, the projector $|i\rangle\langle i|$ is the reducee $(P_i Q)'$ of $P Q$ in $\mathcal{R}(Q)$. Therefore, an equivalent form of the criterion in lemma 6 is completeness (in the absolute sense) of the reducee A' of A in $\mathcal{R}(Q)$.

6. Correlation incompatibility

Now we turn to bipartite states.

Lemma 7. *A subsystem observable $A_1 \otimes 1$ is* discrete *in relation to a state ρ_{12}* if and only if *so is A_1 in relation to ρ_1 $(\equiv \mathrm{Tr}_2 \rho_{12})$.*

Proof. It follows immediately from $\forall i : \quad \mathrm{Tr}\big(P_1^i \rho_{12}\big) = \mathrm{Tr}\big(P_1^i \rho_1\big)$. □

Let us introduce a general concept.

Definition 5. *When a first-subsystem observable A_1 and a bipartite state ρ_{12} are related so that $[A_1, \rho_s] = 0, \ s = 1, 2$ and $[A_1, \rho_{12}] \neq 0$, and the observable is discrete in relation to ρ_1, then we say that we have a case of* correlation incompatibility. *The same term will be used for the symmetric case with a second-subsystem observable A_2.*

Intuitively one expects that in this case the very incompatibility $[A_s, \rho_{12}] \neq 0$, being only due to the correlations in ρ_{12}, should, through its amount, play a role in understanding the correlations contained in $\rho_{12}, s = 1$ or 2. Therefore, we investigate the relation between the coherence entropy $E_C(A_1, \rho_{12})$ and the von Neumann mutual information $I(\rho_{12}) \equiv S(\rho_1) + S(\rho_2) - S(\rho_{12})$.

Theorem 4. *In the case of* correlation incompatibility, *one has*

$$I(\rho_{12}) = E_C(A_s, \rho_{12}) + I\left(\sum_i P_s^i \rho_{12} P_s^i\right) \qquad s = 1 \text{ or } 2 \tag{30}$$

i.e. the coherence entropy is a term in the von Neumann mutual information *of ρ_{12} (together with another possible nonnegative term).*

Proof. One has always (in obvious notation) $S_{12} = S_1 - I_{12} + S_2$. We assume that $s = 1$. In our case $\sum_i P_1^i \rho_1 P_1^i = \rho_1$, hence

$$\begin{aligned}
E_C(A_1, \rho_{12}) &\equiv S\left(\sum_i P_1^i \rho_{12} P_1^i\right) - S(\rho_{12}) \\
&= \left[S\left(\sum_i P_1^i \rho_1 P_1^i\right) - I\left(\sum_i P_1^i \rho_{12} P_1^i\right) + S\left(\mathrm{Tr}_1 \left[\left(\sum_i P_1^i\right)\rho_{12}\right]\right)\right] \\
&\quad - [S(\rho_1) - I(\rho_{12}) + S(\rho_2)] \\
&= I(\rho_{12}) - I\left(\sum_i P_1^i \rho_{12} P_1^i\right)
\end{aligned}$$

because, $\sum_i P_1^i$ being a certain event in ρ_{12}, one has $\left(\sum_i P_1^i\right)\rho_{12} = \rho_{12}$ (cf (6)). If $s = 2$, the proof is symmetrical. □

Thus, the above intuitive expectation turned out to be correct. But it is not clear if the coherence entropy at issue belongs to the amount of quasi-classical correlation or to that of entanglement in ρ_{12}. The latter seems more likely because coherence and incompatibility are unknown in classical physics.

7. Back to physical twin observables

Let us start by generalizing the mentioned result from [3] that a tailor-made pair of physical twin observables (PTO) A_1, A_2 for an arbitrary given bipartite pure state $\rho_{12} \equiv |\Phi\rangle_{12}\langle\Phi|_{12}$ 'carries' the amount of entanglement (quantum discord) via the coherence entropy of A_1 (or of A_2) in $|\Phi\rangle_{12}$.

Lemma 8. *If A_1, A_2 are PTO for $|\Phi\rangle_{12}$ and A_1 is complete in relation to ρ_1, the state (reduced density operator) of subsystem 1, then $E_C(A_1, |\Phi\rangle_{12}) = S(\rho_1)$, and thus these PTO 'carry' the entire amount of entanglement, i.e. the quantum discord in $|\Phi\rangle_{12}$.*

Remark 5. Bearing in mind that for PTO A_1 and ρ_1 are compatible (cf theorem 1) A_1 is complete in relation to ρ_1 if and only if its reducee A_1' in $\mathcal{R}(Q_1)$ (Q_1 being the range projector of ρ_1) is complete in the absolute sense (cf remark 4).

Proof of lemma 8. This is obtained by reducing this case to the 'tailor-made' one from [3]. Namely, the commutation $[A_1, \rho_1] = 0$ enables one to choose the eigen-subbasis of ρ_1 that spans its range as an eigen-subbasis also of A_1. Expansion of $|\Phi\rangle_{12}$ in this basis then gives the Schmidt biorthogonal form, and A_1, A_2 have (the 'tailor-made') spectral forms as in [3]. □

We now go to a *general bipartite state* ρ_{12}.

As is well known, there is no entropy increase in ideal measurement of the observable at issue if and only if the observable and state are compatible. Since for PTO (4) is valid, we have a case of correlation incompatibility if A_1 has a nonzero weak component in relation to ρ_{12}.

Further investigation requires a general result concerning the so-called biorthogonal mixtures of bipartite states. Let us define this concept.

Definition 6. *Let* $\{P_1^k : \forall k\}$ *and* $\{Q_2^k : \forall k\}$ *be any sets of* orthogonal *projectors for the first and the second subsystem, respectively, with common enumeration, and let* $\sum_k p_k \rho_{12}^k$ *be a mixture such that* $\forall k : \rho_{12}^k = P_1^k \rho_{12}^k Q_2^k$. *Then the* mixture *is said to be* biorthogonal.

Now we can prove a general result. It is the analogue of the mixing property of entropy. It can be called the mixing property of von Neumann mutual information.

Lemma 9. *Let* $\rho_{12} = \sum_k p_k \rho_{12}^k$ *be a* biorthogonal mixture. *Then,*

$$I(\rho_{12}) = H(p_k) + \sum_k p_k I\big(\rho_{12}^k\big). \tag{31}$$

Thus the von Neumann mutual information is the sum of the mixing entropy and the average von Neumann mutual information of the component states in the mixture.

Proof. By definition $I(\rho_{12}) = S(\rho_1) + S(\rho_2) - S(\rho_{12})$. The mixture entails $\rho_s = \sum_k p_k \rho_s^k$, $s = 1, 2$, where $\forall k : \rho_s^k \equiv \mathrm{Tr}_{s'} \rho_{12}^k$, $s, s' = 1, 2$, $s \neq s'$. Since, besides the composite mixture, also both subsystem mixtures are orthogonal, one has three mixing properties of entropy: $S(\rho_{12}) = H(p_k) + \sum_k p_k S\big(\rho_{12}^k\big)$, $S(\rho_s) = H(p_k) + \sum_k p_k S\big(\rho_s^k\big)$, $s = 1, 2$. Substituting the three entropy decompositions in the above definition of mutual information, one obtains the claimed relation (31). □

Theorem 5. *If* A_1 *and* A_2 *are* physical twin observables *for* ρ_{12}, *then*

$$I(\rho_{12}) = S(A_s, \rho_{12}) + E_C(A_s, \rho_{12}) + \sum_i p_i I\big(\rho_{12}^i\big) \qquad s = 1, 2 \tag{32a}$$

where

$$\forall i : \quad \rho_{12}^i \equiv P_1^i \rho_{12} P_1^i \big/ p_i = P_2^i \rho_{12} P_2^i \big/ p_i \tag{32b}$$

and $\sum_i a_i P_s^i$ *is the detectable part of* $A_s, \forall i : \quad p_i \equiv \mathrm{Tr}\big(\rho_{12} P_s^i\big)$, $s = 1, 2$. *Thus, both the entropy and the coherence entropy of* A_s *in* ρ_{12} *are parts of the von Neumann mutual information. Besides,* $S(A_1, \rho_{12}) = S(A_2, \rho_{12})$ *and* $E_C(A_1, \rho_{12}) = E_C(A_2, \rho_{12})$.

Proof. The last term in relation (30) is the mutual information of a biorthogonal mixture (cf definition 6 and (5*b*)). Applying the mixing property of mutual information (lemma 9) to it, the claimed relation (32*a*) is immediately derived because $H(p_i) = S(A_s, \rho_{12})$. The entropies of A_1 and A_2 are equal because they are both given by $H(p_i)$. Also the coherence entropies coincide due to (5*b*) (rewritten in (32*b*)) (cf (12) and (3)). □

Theorem 6. *If* A_1 *and* A_2 *are PTO for* ρ_{12} *and* A_s *are* complete *in relation to* ρ_s, $s = 1, 2$, *then*

$$I(\rho_{12}) = S(A_s, \rho_{12}) + E_C(A_s, \rho_{12}) \qquad s = 1, 2. \tag{33}$$

The entropy $S(A_s, \rho_{12})$ *of* A_s *in* ρ_{12} *is the quasi-classical or subsystem-measurement accessible part, and the coherence entropy* $E_C(A_s, \rho_{12})$ *is the* quantum discord, *i.e. the amount of entanglement, in the von Neumann mutual information in* ρ_{12}.

Proof. By assumption we have relative completeness, i.e. $\forall i : \quad Q_s P_s^i = |i\rangle_s \langle i|_s$, where Q_s is the range projector of ρ_s (cf lemma 3 and remark 4), $s = 1, 2$. (Naturally, $\big[Q_s, P_s^i\big] = 0$ is a consequence of (4).) Relations (32*b*) imply

$$\rho_s^i \equiv \mathrm{Tr}_{s'} \rho_{12}^i = (|i\rangle_s \langle i|_s \rho_s |i\rangle_s \langle i|_s)/p_i = |i\rangle_s \langle i|_s \qquad s, s' = 1, 2 \quad s \neq s'.$$

Thus, $\forall i : \quad \rho_{12}^i = |i\rangle_1 \langle i|_1 \otimes |i\rangle_2 \langle i|_2$, hence $I\big(\rho_{12}^i\big) = 0$, and (32*a*) reduces to (33) as claimed.

To obtain the quasi-classical part I_{qcl} of $I(\rho_{12})$, we first evaluate the subsystem entropies, which are upper bounds for the former (cf (4*a*), (4*b*) in [3]). Since $[\rho_s, A_s] = 0$ (cf (4)), ρ_s and A_s have a common eigenbasis in $\mathcal{R}(Q_s)$, $s = 1, 2$. Further, the reducees A'_s in $\mathcal{R}(Q_s)$ are complete (cf remark 4), hence the common eigenbasis in $\mathcal{R}(Q_s)$ is the eigenbasis $\{|i\rangle_s : \forall i\}$ of A'. Then, if $\rho_s = \sum_i r_i^s |i\rangle_s \langle i|_s$, $S(\rho_s) = H\big(r_i^s\big)$. Finally, $\forall i : r_i^s = \langle i|_s \rho_s |i\rangle_s = \operatorname{Tr} \rho_s |i\rangle_s \langle i|_s = \operatorname{Tr} \rho_s \big(P_s^i Q_s\big) = p_i, \quad s = 1, 2$. Hence,

$$S(\rho_s) = H(p_i) = S(A_s, \rho_{12}) \qquad s = 1, 2. \tag{34}$$

If one performs simultaneous measurement of $(A_1 \otimes 1)$ and $(1 \otimes A_2)$ on ρ_{12} (denoted by $(A \wedge A)$), A_1 and A_2 being the PTO in the theorem, then one has a classical discrete joint probability distribution $p_{ii'} \equiv \operatorname{Tr}[\rho_{12}(|i\rangle_1\langle i|_1 \otimes |i'\rangle_2\langle i'|_2)]$, where $|i\rangle_1\langle i|_1 = P_1^i Q_1$, $|i'\rangle_2\langle i'|_2 = P_2^{i'} Q_2$, $A_1 = \sum_i a_i P_1^i + P_1^\perp A_1 P_1^\perp$ (cf (7)) and $A_2 = \sum_{i'} a'_{i'} P_2^{i'} + P_2^\perp A_2 P_2^\perp$. The probability distribution implies the mutual information

$$I(m1 : m2)_{A \wedge A} \equiv H(p_i) + H(p_{i'}) - H(p_{ii'})$$

where on the RHS we have the Shannon entropies $H(p_{ii'}) \equiv -\sum_{ii'} p_{ii'} \ln p_{ii'}$, $H(p_i) \equiv -\sum_i p_i \ln p_i$ and $H(p_{i'}) \equiv -\sum_{i'} p_{i'} \ln p_{i'}$, and $p_i \equiv \sum_{i'} p_{ii'}$, $p_{i'} \equiv \sum_i p_{ii'}$ are the marginal probability distributions.

On account of the crucial PTO property (5*a*) (or rather its adjoint), it is easily seen that $p_{ii'} = \delta_{i,i'} p_i$. Hence, all three Shannon entropies equal $H(p_i) = S(A_s, \rho_{12})$, $s = 1, 2$, and also

$$I(m1 : m2)_{A \wedge A} = S(A_s, \rho_{12}) \qquad s = 1, 2. \tag{35}$$

On the other hand, two chains of information are valid

$$I(m1 : m2)_{A \wedge A} \leqslant I(m1 \to 2) \leqslant \min\{I(\rho_{12}), S(\rho_2)\} \tag{36a}$$

$$I(m1 : m2)_{A \wedge A} \leqslant I(1 \leftarrow m2) \leqslant \min\{I(\rho_{12}), S(\rho_1)\} \tag{36b}$$

(cf (6*b*) and (7*a*), (7*b*) in [3]). Here $I(m1 \to 2)$ is the maximal information that one can gain by measurement on subsystem 1 about subsystem 2, and $I(1 \leftarrow m2)$ is the symmetrical quantity. (For a more precise definition see [3].)

It is seen from (34), (35), and (36*a*), (36*b*) that we have what may be called a common collapse of the two chains:

$$I(m1 : m2)_{A \wedge A} = I(m1 \to 2) = I(1 \leftarrow m2) = S(\rho_2) = S(\rho_1) = S(A_s, \rho_{12}) \qquad s = 1, 2. \tag{37}$$

Following [6], [14] and [3], we define

$$I_{qcl} \equiv I(m1 \to 2) = I(1 \leftarrow m2). \tag{38}$$

Hence,

$$I_{qcl} = S(A_s, \rho_{12}) \qquad s = 1, 2 \tag{39}$$

as claimed.

Following [6], we take the quantum discord

$$\delta(\rho_{12}) \equiv I(\rho_{12}) - I_{qcl} \tag{40}$$

as the measure of entanglement in ρ_{12}. (In [14] a different measure of entanglement is defined. It is independent of I_{qcl}. It coincides with $\delta(\rho_{12})$ for pure states, but not for mixed states in general.)

In view of (40) and (39), one can conclude from (33), which has already been proved, that

$$\delta(\rho_{12}) = E_C(A_s, \rho_{12}) \qquad s = 1, 2 \tag{41}$$

as claimed. □

All pure bipartite states $\rho_{12} = |\Phi\rangle_{12}\langle\Phi|_{12}$ are examples to which theorem 6 applies. But there are also mixed states of this kind. To show this we need an auxiliary result.

Remark 6. If

$$\rho_{12} = \sum_k w_k |\Phi\rangle_{12}^k \langle\Phi|_{12}^k \tag{42}$$

is any decomposition of a bipartite state into pure ones, then opposite-subsystem observables A_1, A_2 are PTO for ρ_{12} *if and only if* they are PTO for each $|\Phi\rangle_{12}^k$. (The detectable eigenprojectors P_s^i are algebraic twin observables, and this statement has been proved for such observables, cf C2 in section 3 of [9].)

Remark 7. To obtain mixed states ρ_{12} to which theorem 6 applies, we define several bipartite pure states via their Schmidt biorthogonal decompositions:

$$\forall k: \quad |\Phi\rangle_{12}^k \equiv \sum_i \left(r_i^k\right)^{1/2} |i\rangle_1 |i\rangle_2 \qquad \forall k, \forall i: \quad r_i^k > 0 \tag{43}$$

(we take $\left\{r_i^k : \forall i\right\}$ distinct for different values of k). Then $A_1 \equiv \sum_i a_i |i\rangle_1\langle i|_1$ and $A_2 \equiv \sum_i b_i |i\rangle_2\langle i|_2$ (a_i and separately b_i distinct) are PTO for each $|\Phi\rangle_{12}^k$, in particular, *complete* ones in relation to the corresponding subsystem states of $|\Phi\rangle_{12}^k$ (because the eigenvalues are distinct). Hence, they are *complete* in relation to the corresponding subsystem states of ρ_{12} defined by (42) (cf remarks 6 and 4). Therefore, theorem 6 applies to this case.

Let us return to theorem 5, where A_1, A_2 are PTO but not necessarily complete in relation to ρ_1, ρ_2. $S(A_s, \rho_{12})$ $s = 1, 2$ is certainly a part of I_{qcl} (as clear from the argument in the proof of theorem 6). Hopefully, also $E_C(A_s, \rho_{12})$, $s = 1, 2$ is a part of the quantum discord $\delta(\rho_{12})$.

8. Conclusion

Three groups of results have been obtained in this study.

(i) The concept of physical twin observables is made more practical by simplifying it in theorem 1.

(ii) Utilizing well-known relations, a somewhat surprising general relation between the coherence entropy $E_C(A, \rho)$ and the entropy $S(A, \rho)$ of A in ρ was established (relation (16), inequality (17) and figure 1). Further, it was shown that the coherence entropy satisfies the third intuitive requirement (see the introduction and lemma 1). This property led to another form of $E_C(A, \rho)$, in which the redundancies are omitted (theorem 2). Finally, it was proved that if one considers a refinement A' of A instead of the latter, the coherence entropy cannot decrease. A sufficient and necessary condition is given for the increase.

(iii) The case of correlation incompatibility (definition 5) comprises all bipartite pure states and some mixed ones. The role of the coherence entropy $E_C(A_s, \rho_{12})$ of the subsystem observable A_s ($s = 1$ or 2) at issue in the von Neumann mutual information

$I(\rho_{12})$ is investigated in this case, and three, more and more specific, results are obtained:

(a) $I(\rho_{12})$ is the sum of $E_C(A_s, \rho_{12})$ and a possible nonnegative term (theorem 4).

(b) If A_s is one of physical twin observables in relation to ρ_{12}, then $I(\rho_{12})$ is the sum of the entropy $S(A_s, \rho_{12})$ of the observable A_s in ρ_{12} and $E_C(A_s, \rho_{12})$ and a possible nonnegative term (theorem 5).

(c) If the twin observables in (b) are complete in relation to ρ_{12}, then the third term mentioned in (b) is necessarily zero (theorem 6 and relation (33)). A bipartite mixed-state example is given (remark 7).

The result in (c) may be of significance for the important problem of how to split $I(\rho_{12})$ into a quasi-classical part and a part that is the amount of purely quantum entanglement. Namely, the term $S(A_s, \rho_{12})$ can be pretty safely interpreted as precisely the quasi-classical part. In [6, 14], this was defined and in both the entropy $S(\rho_s)$ of the corresponding subsystem state ρ_s is an upper bound for this quantity. On account of the completeness of the PTO, the entropy $S(A_s, \rho_{12})$ equals $S(\rho_s)$; hence, having reached its upper bound, it must be the quasi-classical part of the quantum correlations in ρ_{12}. The quantum discord, the term in $I(\rho_{12})$ that is the excess over the quasi-classical part, is equal to $E_C(A_s, \rho_{12})$ in the case discussed. Thus, the PTO 'carry' both the quasi-classical part (as their entropy in ρ_{12}) and the quantum discord (as their coherence entropy).

Appendix A

Proof of the equivalence of $PQ = Q$ (Q being the range projector of ρ), cf (13), with $\mathrm{Tr}(P\rho) = 1$:

Sufficiency: Since always $\rho = Q\rho$, one has $\mathrm{Tr}(P\rho) = \mathrm{Tr}[P(Q\rho)] = \mathrm{Tr}(Q\rho) = \mathrm{Tr}\,\rho = 1$.

Necessity: Let $\rho = \sum_n r_n |n\rangle\langle n|$ be a spectral form of ρ with positive eigenvalues. Then, on account of (6), $\mathrm{Tr}(P\rho) = 1$ amounts to $P\big[\sum_n (r_n|n\rangle\langle n|)\big] = \sum_n (r_n|n\rangle\langle n|)$. Multiplying this from the right by $|n'\rangle\langle n'|$ with a fixed n' value, taking the trace and dividing by $r_{n'}$, one obtains $\mathrm{Tr}(P|n'\rangle\langle n'|) = 1$. Utilizing (6) again, one has $P|n'\rangle\langle n'| = |n'\rangle\langle n'|$. Finally, since $Q = \sum_{n'} |n'\rangle\langle n'|$, the claimed relation (13) ensues.

Appendix B

Proof of the claim that if the density operator ρ is nonsingular, then only the zero event $P = 0$ has zero probability in ρ:

$$\mathrm{Tr}(P\rho) = 0 \quad\Rightarrow\quad \mathrm{Tr}(P^{\perp}\rho) = 1 \quad\Rightarrow\quad P^{\perp}Q = Q$$

(cf appendix A). But now $Q = 1$. Hence, $P^{\perp} = 1$, and $P = 0$.

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